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# ANALYTICAL SOLUTIONS OF THE ADVECTION-DIFFUSION EQUATION

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**ABSTRACT**

The paper provides a review air quality mathematical modeling of the advection diffusion equation with analytical solutions. In advection diffusion equation, we write the wind velocity vector and concentration field as the sum of the average velocity $(\bar{u}, \bar{C})$ plus the fluctuating (turbulent) components$(u', C')$. First consider the dispersion of an air pollutant in the atmosphere under steady state condition by neglecting the diffusion in downwind direction, then we present solution of advection diffusion equation when dispersion of an air pollutant with constant wind velocity and constant removal rate. We also described Gaussian plume model with solution for the variation of concentration of air pollutants and solution of advection diffusion equation for variable and constant eddy diffusivity and wind speed.

***Keywords:*** Eddy Diffusivity, Advection Diffusion Equation, Analytical Solutions, Steady-State, Wind Velocity.

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## 1. INTRODUCTION

In air quality mathematical modeling, analytical solutions of equations give understanding and description of physical phenomena considering all the parameters of a problem, so that their influence can be reliably investigated and it easy to obtain the asymptotic behaviour of the solution, which is usually difficult to generate through numerical calculations. The first solution of the advection-diffusion equation was the well known Gaussian solution discovered by Fick in the mid- 19th century. In the Gaussian solution both wind and eddy diffusion coefficients are constant with height and the boundary conditions are:

$$K_z \frac{\partial \bar{C}}{\partial z} = 0 \;\; at\ z = 0\ and\ z = \infty$$

They are usually the boundary conditions of the analytical solutions of the advection-diffusion equation and are assumed as such below, unless others are specified. Roberts [5] presented a bidimensional solution of Eq. (9), for ground level sources only, in cases where both wind speed $u$ and vertical diffusion coefficients $K_z$ follow power laws as a function of height . That is: $u = u_1 \left( {}^{z}/_{z_1} \right)^{\alpha}$ and $K_z = K_1 \left( {}^{z}/_{z_1} \right)^{\beta}$. Where $u_1$ and $K_1$ are evaluated and $z_1$being the height. Rounds [6] obtained a bidimensional solution valid for elevated sources with the above wind profile, but only for linear profiles of $K_z$. Chrysikopoulos et al. [1] presented a three dimensional solution for the transport of non-buoyant emission from a continuous ground level area source for the same profiles of $u$ and $K_z$, but including dry deposition as a removal mechanism. Smith [7] resolved the bidimensional equation of transport and diffusion with $u$ and $K_z$ power functions of height, with the exponents of these functions following the conjugate law of Schmidt ($\alpha = 1 - \beta$).

In this paper we present a short review of analytical solutions of the advection-diffusion equation. First, we consider the dispersion of an air pollutant emitted from an elevated point source is investigated and analyzed by taking constant wind velocity and constant removal rate [10]. The eddy diffusivities are also taken as constants. We also described Gaussian plume model

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for the variation of concentration of air pollutants, $C$, from an elevated source in presence of wind, in steady state (Stockie, 2011) [8].

## 2. GOVERNING EQUATIONS

The advection diffusion equation of air pollution in the atmosphere is essentially a statement of conservation of the suspended material and can be written as follows [10]:

$$\frac{\partial C}{\partial t} = -u.\nabla C + D\nabla^2 C + S$$

(2.1)

where $u$ is the wind velocity vector, with components $u, v, w$; $D\nabla^2 C$ is the molecular diffusion term, where $D$ is the molecolar diffusion coefficient; $C$ is the concentration of the conserved material; $S$ is the source term giving the emission flux and representing the removal kinetics of the pollutant; $\nabla$ is the gradient operator and $\nabla^2$ is the Laplacian operator.

If we write the wind velocity vector and concentration field as the sum of the average velocity $(\bar{u}, \bar{C}\,)$ plus the fluctuating (turbulent) components$(u', C')$, we have:

$$u = \bar{u} + \bar{C}$$

(2.2)

$$C = u' + C'$$

(2.3)

Therefore, Eq. (2.1) is written as

$$\frac{\partial \bar{C}}{\partial t} = -\bar{u}.\nabla\bar{\mathrm{C}} - \nabla.\overline{\mathrm{C'u'}} + D\nabla^2 C + \bar{S}$$

(2.4)

where $\overline{\mathrm{C'u'}}$ is turbulent concentration flux. The simplest closure method of Eq. (2.4) is a local first order closure in which K-theory is used. Assuming that turbulent concentration flux is proportional to the gradient of the average concentration, the following relation is obtained as:

$$\overline{\mathrm{C'u'}} = -\mathrm{K}\nabla\bar{\mathrm{C}}$$

(2.5)

where K (3x3) is turbulent diffusion coefficient. When K tensor is diagonal, molecular diffusion is negligible and $C(x, y, z, t)$ represents the concentration of a non-reactive pollutant ($\bar{S} = S$). Thus, the Eq. (2.4) can be written as

$$\frac{\partial \bar{C}}{\partial t} = -\bar{u}.\nabla\bar{C} - \nabla.K\,\nabla\bar{C} + S$$

(2.6)

The dispersion of pollutants in the atmosphere is governed by the basic atmospheric diffusion equation. Under the assumption of incompressible flow, atmospheric diffusion equation based on the Gradient transport theory Eq.(2.6) can be written in the rectangular coordinate system as:

$$\frac{\partial C}{\partial t} + u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} + w\frac{\partial C}{\partial z} = \left[\frac{\partial}{\partial x}\left(K_x\frac{\partial C}{\partial x}\right) + \frac{\partial}{\partial y}\left(K_y\frac{\partial C}{\partial y}\right) + \frac{\partial}{\partial z}\left(K_z\frac{\partial C}{\partial z}\right)\right] + S \qquad (2.7)$$

where $C$ is the mean concentration of a pollutant and $S$ is the source term, respectively; $(u, v, w)$and $(K_x, K_{y,}K_z)$ are the components of wind and diffusivity vectors in $x, y\ and\ z$ directions, respectively.

The following assumptions are made in order to simplify Equation (2.7):

1) Steady-state conditions are considered, $i.e.\frac{\partial C}{\partial t} = 0$.

2) As the vertical velocity is much smaller than the horizontal one in $x$-direction, the term $w\frac{\partial C}{\partial z}$ is neglected.

3) The $x$-axis is oriented in the direction of mean wind $u = U$ and $U$ much greater than the wind speed $v$ in $y$ -direction the term $v\frac{\partial C}{\partial y}$ is neglected. With the above assumptions, Equation (1) reduces to:

$$U\frac{\partial C}{\partial x} = \left[\frac{\partial}{\partial x}\left(K_x\frac{\partial C}{\partial x}\right) + \frac{\partial}{\partial y}\left(K_y\frac{\partial C}{\partial y}\right) + \frac{\partial}{\partial z}\left(K_z\frac{\partial C}{\partial z}\right)\right] + S$$

(2.8)

5) The advection term in $x$ direction is larger than the diffusion in $x$ direction then we will neglect the diffusion term in $x$ direction,

$$U\frac{\partial C}{\partial x} = \left[\frac{\partial}{\partial y}\left(K_y\frac{\partial C}{\partial y}\right)+\frac{\partial}{\partial z}\left(K_z\frac{\partial C}{\partial z}\right)\right]+S$$

(2.9)

6) The eddy diffusivities are functions of the downwind distance $x$ only, and diffusion is isotropic so that $K_x = K_y = K_z = K$, equation (2.9) reduces to following partial differential equation (Stockie, 2011) [10].

$$U\frac{\partial C}{\partial x} = K\left(\frac{\partial^2 C}{\partial y^2}+\frac{\partial^2 C}{\partial z^2}\right)+S$$

(2.10)

The contaminant is emitted at a constant rate $Q$ [kg/s] from a single point source $x = (0,0,H)$ located at height $H$ above the ground surface, as depicted in Figure 2.1. Then the source term may be written as

$$S = Q\delta(x)\delta(y)\delta(z-H)$$

(2.11)

where $\delta(.)$ is the Dirac delta function. Note that the units of the delta function are [$m^{-1}$]. For the stack-like configuration pictured in Figure 2.1 the height is actually an *effective height* $H = h+\delta h$, which is the sum of the actual stack height $h$ and the *plume rise* $\delta h$ that arises from buoyant effects. Then the equation (2.10) reduces to following partial differential equation

$$U\frac{\partial C}{\partial x} = K\left(\frac{\partial^2 C}{\partial y^2}+\frac{\partial^2 C}{\partial z^2}\right)+Q\delta(x)\delta(y)\delta(z-H)$$

(2.12a)

The boundary conditions for the model equations are:

$$C(0,y,z) = 0 \qquad (2.12b)$$

$$C(\infty,y,z) = 0 \qquad (2.12c)$$

$$C(x,\pm\infty,z) = 0 \qquad (2.12d)$$

$$C(x,y,\infty) = 0 \qquad (2.12e)$$

$$K\frac{\partial C}{\partial z}(x,y,0) = 0 \qquad (2.12f)$$

The condition (2.12b) is a consequence of the unidirectional wind and the assumption that there are no contaminant sources for $x < 0$. The conditions(2.12c), (2.12d), and (2.12e) respectively assume that concentration, $C$ decays to zero as $x$ tends to $\infty, y$ tends to $\pm\infty$ and flux is zero at the earth's surface and according to assumption that the contaminant does not penetrate the ground, the vertical flux at the ground must vanish, which leads to the boundary condition(2.12f). This equation can be solved by the method of separation of variables. Here, we have formulated a solution given by Stockie (2011).

When taken together, (2.12b)-(2.12f) represent a well-posed problem for the steady state contaminant concentration $C(x, y, z)$. An equivalent formulation of this problem can be found by eliminating the source term from the PDE and instead introducing a delta function term into the boundary condition [10]:

$$U\frac{\partial C}{\partial x} = K\frac{\partial^2 C}{\partial y^2} + K\frac{\partial^2 C}{\partial z^2} \quad (2.13a)$$

$$C(0, y, z) = \frac{Q}{U}\delta(y)\delta(z - H) \quad (2.13b)$$

$$C(\infty, y, z) = 0 \quad (2.13c)$$

$$C(x, \pm\infty, z) = 0 \quad (2.13d)$$

$$C(x, y, \infty) = 0 \quad (2.13e)$$

$$K\frac{\partial C}{\partial z}(x, y, 0) = 0 \quad (2.13f)$$

The equivalence between problems (2.12) and (2.13) for $x > 0$ is presented. It is the second form of the governing equations that will be used in deriving the analytical solution.

## 3. MATHEMATICAL MODELING FOR DISPERSION OF AN AIR POLLUTANT WITH CONSTANT WIND VELOCITY AND CONSTANT REMOVAL RATE

The dispersion of an air pollutant in the atmosphere under steady state condition by neglecting the diffusion in downwind direction, is described by partial differential equation [10]

$$U\frac{\partial C}{\partial x} = K_y\frac{\partial^2 C}{\partial y^2} + K_z\frac{\partial^2 C}{\partial z^2} - \alpha C$$

(3.1)

where $U$ is the wind velocity taken to be constant, $C$ is the concentration of the air pollutant, $K_y$ and $K_z$ are the eddy diffusivities in $y$- and $z$ directions respectively which are assumed constants, and a is the removal rate of the air pollutant due to some natural mechanism like chemical reaction, which is also taken to be constant. Typically $K_y > K_z$ in the atmosphere.

An analytical approach to the problem of dispersion of an air pollutant with constant wind velocity and constant removal rate is proposed to study. Eddy diffusivities are also taken as constant. It is shown that the concentration profile of an air pollutant decreases continuously with increasing downwind distance while it increases with increasing vertical distance. This increase in concentration with vertical distance is higher at lower values of downwind distance and there is negligible change in the concentration of air pollutant with increasing vertical distance at higher values of downwind distances. Where the dispersion of an air pollutant emitted from an elevated

point source is investigated and analyzed by taking constant wind velocity and constant removal rate. The eddy diffusivities are also taken as constants.

The boundary conditions for the equation (3.1) are taken as follows:

$$C(x,y,z) = \frac{Q\delta(y)\delta(z-h_s)}{U}, \qquad x = 0, 0 < h_s < H, \tag{3.2}$$

$$C(x,y,z) = 0, \quad y \to \pm\infty, \tag{3.3}$$

$$C(x,y,z) = 0, \ \ z = 0, \tag{3.4}$$

$$K_z \frac{\partial C}{\partial z} = v_d C, Z = H, \tag{3.5}$$

where $\delta$ is the Dirac delta-function, $Q$ is the emission strength of an elevated point source, $h_s$ is the stack height and $v_d$ is the deposition velocity of the air pollutant. Condition (3.2) states that the air pollutant is released from the elevated point source of strength $Q$. Condition (3.3) states that the concentration of the air pollutant is zero for $y \to \pm\infty$ Condition (3.4) states that the concentration of the air pollutant is zero at the ground surface and condition (3.5) states that there is some diffusion flux at the vertical height $H$ from the ground surface.

First of all, the partial differential equation (3.1) describing the dispersion of the air pollutant and the boundary conditions are made non-dimensional by introducing the non-dimensional quantities. Then applying Fourier transform technique and method of separation of variables solve the dispersion equation in steady state condition. We get [10]

$$C = \left\{\frac{0.28204}{\sqrt{\beta x U}}\right\} \exp\left[-\left\{\frac{\alpha x}{U} + \frac{y^2 U}{4\beta x}\right\}\right] \sum_{n=1}^{\infty} \left\{\frac{\sin(\lambda_n z)\sin(\lambda_n h_s)}{P_n}\right\} exp\left\{-\left(\frac{\gamma x}{U}\right)\right\} \lambda_n^2,$$

(3.6)

Where $P_n = \int_0^1 sin^2(\lambda_n z) dz.$

**4. GAUSSIAN PLUME MODEL**

The variation of concentration of air pollutants, *C*, from an elevated source in presence of wind, in steady state, is described by the following partial differential equation. Stockie (2011) [8] has given a detailed analysis of mathematics given solution of following equation.

$$U\frac{\partial C}{\partial x} = K\left(\frac{\partial^2 C}{\partial y^2} + \frac{\partial^2 C}{\partial z^2}\right)$$

(4.1)

Where $U$ the wind speed and K is is the diffusion coefficient. Here, wind direction is in *x*-direction which is horizontal, *y* is horizontal and perpendicular to *x*, and *z*-direction is vertical increasing upwards. The source of pollutant having strength as *Q* is located at coordinates: (0,0,*H*). This source is represented in terms of Delta function as [4]

$$C(0, y, z) = Q\delta(x)\delta(y)\delta(z - H)$$

(4.2)

The boundary conditions for the model equations are:

$$C(x, \pm\infty, z) = 0$$

(4.3)

$$C(x, y, \infty) = 0$$

(4.4)

$$K\frac{\partial C}{\partial z}(x, y, 0) = 0$$

(4.5)

These conditions respectively assume that concentration, $C$ decays to zero as $x$ tends to $\infty, y$ tends to $\pm\infty$ and flux is zero at the earth's surface. We have all necessary boundary conditions for the air quality equation. This equation can be solved by the method of separation of variables. Stockie (2011) has given a detailed analysis of mathematics of this solution. The solution for concentration $C(x, y, z)$, called Gaussian plume solution, is given as

$$C(x, y, z) = \frac{Q}{4\pi Kx} e^{-\frac{y^2U}{4Kx}} \left[ e^{-\left(\frac{U(z-h)}{4Kx}\right)^2} + e^{-\left(\frac{U(z+h)}{4Kx}\right)^2} \right]$$

(4.6)

This equation is made of simple exponential functions. Each exponential function is Gaussian type, like $e^{-p^2}$having value as one at $p = 0$ and decaying to zero as $p$ tends to infinity. This solution can be used to build solution for various sources located at various locations as the air quality equation given above is linear and principle of superposition can be used. Stockie (2011) has presented several numerical results. This model can be used both for physical understanding and also regulations.

## 5. SOLUTION FOR CONSTANT EDDY DIFFUSIVITY AND WIND SPEED

The solution of equation (3) has been presented by [3] by solving advection diffusion equation analytically using separation of variables technique, considering first the wind speed and eddy diffusivity as constants; second as variables dependent on vertical height *z*.

First, consider the wind speed and eddy diffusivity as constants. Use Equation (3) and consider the wind speed $U$ and eddy diffusivity $z\ k$ as constant. We get solution

$$C(x, z) = \frac{Q}{Uz_s} e^{-\lambda^2 x} \cos(\lambda^2 z) \sec(\lambda^2 z_s)$$

(5.1)

## 6. SOLUTION FOR VARIABLE EDDY DIFFUSIVITY AND WIND SPEED

Secondly, consider variables Eddy Diffusivity and Wind Speed. Where the variables are depend on vertical height $z$. Consider the wind speed $U$ as linear of $z$:

$$U = k_0 u_* z, z \neq 0 \; and \; U = U_0 \; at \; z = 0$$

(6.1)

and eddy diffusivity $k_z$ is expressed as functions of power law of $z$ as:

$$k_z = u_1 z^n$$

(6.2)

where $k_0$ is Von-Karmen constant and $u_*$ is the friction velocity. Where $u_1$ is turbulence intensity and given a detailed analysis of mathematics given solution.

$$C_y(x, y) = Q_p \; \frac{2(zz_s)^{1-n}}{k_0 u_* h^2} \sum_{\alpha=1}^{\infty} \frac{J_\mu\left(\eta_\alpha z^{\frac{3-n}{2}}\right) J_\mu\left(\eta_\alpha z_s^{\frac{3-n}{2}}\right)}{J_{\mu+1}^2\left(\eta_\beta h^{\frac{3-n}{2}}\right)} \exp(-\lambda^2 x)$$

(6.3)

Conclusion: In Dispersion of an Air Pollutant with Constant Wind Velocity and Constant Removal Rate, increase in concentration with vertical distance is higher at lower values of downwind distance and there is negligible change in the concentration of air pollutant with increasing vertical distance at higher values of downwind distances, Solution of Gaussian plume model is simple exponential functions. Each exponential function is Gaussian type, like $e^{-p^2}$ having value as one at $p = 0$ and decaying to zero as $p$ tends to infinity and when the eddy diffusivity and the wind speed are assumed to be constant times and variable times. It is seen that the concentration profile of air pollutant becomes high near the ground but as the distance from the ground increases, the concentration profile decreases regularly.

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